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Hankel Operators and Toeplitz Operators on the Bergman Space*

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We consider in this paper the question of when the semi-commutator $T_{fg} - T_f T_g$ on the Bergman space with bounded harmonic symbols is compact. Several conditions equivalent to compactness of $T_{fg} - T_f T_g$ are given. As a consequence we prove a conjecture of Axler that for bounded analytic functions f and g on the unit disk, $T_f^* T_g - T_g T_f^*$ is compact iff either f or g is constant on each Gleason part $P(m)$ except D.

We consider in this paper the question of when the product $H_f^* H_g$ of two Hankel operators on the Bergman space with bounded harmonic symbols is compact. The product $H_f^* H_g$ is equal to the semi-commutator $T_{fg} - T_f T_g$. Several conditions equivalent to compactness of $H_f^* H_g$ are given. Consequently we prove Axler's conjecture [2].

As is well known, for f and g in $L^\infty(\partial D)$, Axler, Chang, and Sarason [3] and Volberg [8] have shown that $H_f^* H_g$ on the Hardy space is compact iff $H^\infty[f] \cap H^\infty[g] \subset H^\infty + C(\partial D)$. By means of the theorem of Axler and Shields [5], we also obtain that $H_f^* H_g$ is compact iff $H^\infty[f] \cap H^\infty[g] \subset \{u \in C(\mathscr{M}) : u|_{P(m)} \in H^\infty|_{P(m)}$ for thin part $P(m)$ in $\mathscr{M}\}$ for bounded harmonic functions f and g.

Let D denote the open unit disk in the complex plane $\mathbb{C}$, and dA the usual normalized area measure on D. The Bergman space L_a^2 is the Hilbert space of analytic functions $g: D \to \mathbb{C}$ with inner product given by

$$\langle f, g \rangle = \int_D f(z)\, \bar{g}(z)\, dA(z).$$

As usual, $L^\infty(D)$ denotes the set of bounded measurable functions on D, and $H^\infty(D)$ is the set of bounded analytic functions on D. Let P denote the

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orthogonal projection of $L^2(D, dA)$ onto $L_a^2(D)$. For $f \in L^\infty(D)$, the Hankel operator $H_f: L_a^2 \to (L_a^2(D))^\perp$ and the Toeplitz operator $T_f: L_a^2 \to L_a^2$ are defined by $H_f(h) = (I - P)(fh)$ and $T_f(h) = P(fh)$, respectively.

Let $\mathscr{M}$ be the maximal ideal space of $H^\infty(D)$. The Gleason part $P(m)$ corresponding to m is the equivalence class of a point m in $\mathscr{M}$,

$$P(m) = (m_1 \in \mathscr{M}: \rho(m, m_1) < 1\},$$

where $\rho(m, m_1)$ is the pseduo-hyperbolic distance from m_1 to m defined by

$$\rho(m, m_1) = \sup\{|f(m_1)|; f \in H^\infty(D), \|f\|_\infty \leqslant 1 \text{ and } f(m) = 0\}.$$

If m and m_1 are in the usual disk, the pseduo-hyperbolic distance is given by

$$\rho(m, m_1) = \left|\frac{m_1 - m}{1 - \bar{m}_1 m}\right|.$$

If α is a point of D, let $L_\alpha(z)$ be the linear fractional map

$$L_\alpha(z) = \frac{z + \alpha}{1 + \bar{\alpha} z}.$$

Call a sequence (z_n) in D thin if $\lim_{n \to \infty} \pi_{k \neq n} |z_n - z_k| / |1 - \bar{z}_n z_k| = 1$ and a part $P(m)$ thin if m is in the closure of some thin sequence.

Now we state some of Hoffman's results [7] that will be used in this paper.

H1. Let m be any point of $\mathscr{M} \backslash D$. There exists a sequence $\{\beta_n\}$ in D such that $\{\beta_n\}$ has no accumulation points in D, m is in the closure of $\{\beta_n\}$, the coresponding maps L_{β_n} converge pointwise to L_m, a map from D into $\mathscr{M}$, and for any bounded analytic function h, $h \circ L_{\beta_n}$ converges to $h \circ L_m$ uniformly on compacta, so that

$$(h \circ L_m')(0) = \lim_{\beta_n \to m} (1 - |\beta_n|^2)\, h'(\beta_n).$$

H2. If the Gleason part $P(m)$ contains at least two points, $P(m)$ is an analytic disk, and L_m is a one-to-one analytic map from D onto the Gleason part $P(m)$.

The map L_m plays an important role in our paper. The Gleason part does the same job on the Bergman space as the support set on the Hardy space.

For f analytic on D, the Bloch norm $\|f\|_\beta$ of f is defined by

$$\|f\|_\beta = \sup((1 - |\lambda|^2)|f'(\lambda)| : \lambda \in D.$$

The Bloch space β is the set of analytic functions f on D such that $\|f\|_\beta < \infty$.

$D(z, r)$ will denote the pseduo-hyperbolic disc $\{w \in D: \rho(w, z) < r\}$ for $z \in D$ and $0 < r < 1$, and k_z is the normalized Bergman reproducing kernel

$$\frac{1 - |z|^2}{(1 - \bar{z}w)^2}.$$

The following theorem is our main result.

THEOREM. *Suppose f and g are bounded harmonic functions on D. Then the following conditions are equivalent*:

(a) $H_f^* H_g$ *is compact*;

(b) $T_f T_g - T_{fg}$ *is compact*;

(c) *For each thin part $P(m)$ except D, either $f|_{P(m)} \in H^\infty(D)|_{P(m)}$ or $g|_{P(m)} \in H^\infty(D)|_{P(m)}$*;

(d) *For m in $\mathcal{M}\backslash D$, either $f \circ L_m \in H^\infty$ or $g \circ L_m \in H^\infty$*;

(e) $H^\infty(D)[f] \cap H^\infty(D)[g] \subset \{u \in C(\mathcal{M}): u|_{P(m)} \in H^\infty(D)$ *for each thin part $P(m)$ except $D\}$*;

(f) $\lim_{|z| \to 1} \min\{(1 - |z|^2)|(\partial f/\partial \bar{z})(z)|, (1 - |z|^2)|(\partial g/\partial \bar{z})(z)|\} = 0.$

In particular, the following theorem, which was conjectured by Axler [2], is valid.

THEOREM. *Suppose f and g are bounded analytic functions on D. Then the following conditions are equivalent*:

(a′) $H_{\bar{f}}^* H_{\bar{g}}$ *is compact*;

(b′) $T_f T_g^* - T_{\bar{g}f}$ *is compact*;

(c′) *For each thin part $P(m)$ except D, either $f|_{P(m)}$ or $g|_{P(m)}$ is constant*;

(d′) *For each Gleason part $P(m)$ except D, either $f|_{P(m)}$ or $g|_{P(m)}$ is constant*;

(e′) $H^\infty(D)[\bar{f}] \cap H^\infty(D)[\bar{g}] \subset \{u \in C(\mathcal{M}): u|_{P(m)} \in H^\infty(D)|_{P(m)}$ *for each thin part except $D\}$*;

(f′) $\lim_{|z| \to 1} \min\{(1 - |z|^2)|f'(z)|, (1 - |z|^2)|g'(z)|\} = 0.$

We shall prove (d) ⇒ (f) and (f) ⇒ (a) in Section 1 and (a) ⇒ (d) ⇒ (c) ⇔ (e) and (c) ⇒ (a) in Section 2. It is easy to show that

$$T_f T_g - T_{fg} = -H_{\bar{f}}^* H_g.$$

Hence the equivalence of (a) and (b) is true. The equivalence (f′)$\Leftrightarrow$(c′) that may have been known before will be proved in Section 1. In Section 2 we show that if f and g are bounded harmonic functions and $H_f^* H_g = 0$, then either f or g is in $H^\infty(D)$. This means that $T_f^* T_g = T_{\bar{f}g}$ iff either f or g is analytic.

The reader may be enthralled by the special concept, thin part. In fact, it is natural that the thin part plays the special role since there is a function ϕ in H^∞ such that $\phi \circ L_m(z) = z$ for the thin part m, which is not true for all Gleason parts.

At the same time that the results of this paper were obtained, S. Axler and P. Gorkin [4] proved the same result as in Theorem 6, using methods different from ours.

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We first want to describe the function properties of a bounded analytic function f if f is constant on some Gleason part $P(m)$. For convenience we assume $m \notin D$ from now on.

LEMMA 1. *If f is H^∞ and constant on $P(m)$, then for fixed* $0 < r < 1$

$$\lim_{z \to m} \max_{w \in \bar{D}(z, r)} (1 - |w|^2) f'(w) = 0.$$

Proof. Suppose that there is a net (z_α) in D such that $z_\alpha \to m$. Then $f \circ L_{z\alpha} \to f \circ L_m$ uniformly on compacta from H1. Since $f \circ L_m$ is constant, $f \circ L_m'(z) = \lim (f \circ L_{z\alpha})'(z) = 0$.

Since for w in $\bar{D}(z_\alpha, r)$ there is a z in $\bar{D}(0, r)$ such that $w - L_{z_\alpha}(z)$, we get

$$\max_{\bar{D}(z_\alpha, r)} f'(w)(1 - |w|^2) = \max_{\bar{D}(0, r)} |(f \circ L_{z_\alpha})'(z)|(1 - |z|^2) \to 0.$$

Thus we have proved the lemma.

LEMMA 2. *Let f and g be in $H^\infty(D)$. If either f or g is constant on each thin part then for all* $0 < r < 1$

$$\lim_{|z| \to 1} \min \{ \max_{s \in \bar{D}(z, r)} (1 - |s|^2) |f'(s)|, \max_{t \in \bar{D}(z, r)} (1 - |t|^2) |g'(t)| \} = 0.$$

Proof. Suppose either f or g is constant on each thin part, but there are points (z_n) in D with $|z_n| \to 1$ such that for some $\varepsilon > 0$ and fixed $0 < r < 1$

$$\min\{ \max_{s \in \bar{D}(z_n, r)} (1 - |s|^2) |f'(s)|, \max_{t \in \bar{D}(z_n, r)} (1 - |t|^2) |g'(t)| \} \geqslant \varepsilon \qquad \forall n. \quad (*)$$

Clearly z_n may be chosen so that $\{z_n\}$ is a thin sequence.

Let m be in the closure of $\{z_n\}$ in $\mathcal{M}$. There is a subnet $\{z_{nk}\}$ of $\{z_n\}$ converging to m in $\mathcal{M}$; then m is a thin part. Without loss of generality we may assume that f is constant on $P(m)$. It follows from Lemma 1 that

$$\lim_{z_{n_k} \to m} \max_{s \in \bar{D}(z_{n_k}, r)} (1-|s|^2) f'(s) = 0.$$

The above equation contradicts (∗), so the proof is complete.

THEOREM 1. *Suppose f and g are in $H^\infty(D)$. Then the following are equivalent*:

(c′) *either f or g is constant on each thin part $P(m)$*;

(f′) $\lim_{|z| \to 1} \min\{(1-|z|^2)|f'(z)|, (1-|z|^2)|g'(z)|\} = 0.$

Proof. In fact Lemma 2 implies that

$$\lim_{|z| \to 1} \min\{(1-|z|^2)|f'(z)|, (1-|z|^2)|g'(z)|\} = 0,$$

provided that either f or g is constant on each thin part $P(m)$ except D. Suppose that a thin part $m \in \mathcal{M} \backslash D$ and neither f nor g is constant on $P(m)$. From H1 we may assume that $f \circ L_m'(0) \neq 0$ and $g \circ L_m'(0) \neq 0$. Thus there is a net $\{z_n\}$ in D coverging to m such that

$$\lim_{z_n \to m} (1-|z_n|^2) f'(z_n) = f \circ L_m'(0)$$
$$\lim_{z_n \to m} (1-|z_n|^2) g'(z_n) = g \circ L_m'(0).$$

Clearly z_n may be chosen so that $\{z_n\}$ is a thin part.
So

$$\lim_{|z| \to 1} \min\{(1-|z|^2)|f'(z)|, (1-|z|^2)|g'(z)|\}$$
$$= \min\{|f \circ L_m'(0)|, |g \circ L_m'(0)|\} > 0.$$

The above contradiction completes the proof.

LEMMA 3. *Let $0 < r < 1$ and let f and g be functions in the Bloch space. If*

$$\lim_{|z| \to 1} \min\{\max_{s \in \bar{D}(z, r)} (1-|s|^2)|f'(s)|, \max_{t \in \bar{D}(z, r)} (1-|t|^2)|g'(t)|\} = 0$$

then

$$\lim_{|z| \to 1} \int_{D(z, r)} |f(w) - f(z)| |g(w) - g(z)| |k_z|^2 \, dA(w) = 0.$$

Proof. For $w \in D(z, r)$ we have

$$f(w)-f(z)=\int_0^1 f'[tw+(1-t)z](w-z)\,dt.$$

Thus

$$|f(w)-f(z)| \leqslant |w-z| \int_0^1 |f'[tw+(1-t)z]|\,dt \leqslant \max_{\bar{D}(z,r)} |f'(s)|\,|w-z|. \tag{1}$$

Because

$$|w-z| \leqslant \operatorname{diam} D(z,r) \leqslant C \inf_{D(z,r)} (1-|s|^2) \tag{2}$$

it follows from (1) that

$$|f(w)-f(z)| \leqslant C \max_{\bar{D}(z,r)} (1-|s|^2)|f'(s)|.$$

In fact the above inequality is also true if f is replaced by g. Thus

$$\int_{D(z,r)} |f(w)-f(z)||g(w)-g(z)||k_2|^2\,dA(w)$$

$$\leqslant C^2 \int_{D(z,r)} [\max_{s\in \bar{D}(2,r)} (1-|s|^2)|f'(s)|][\max_{t\in \bar{D}(2,r)} (1-|t|^2)|g'(t)|]|k_2|^2\,dA$$

$$\leqslant C^2 \min\{\max_{s\in \bar{D}(z,r)} (1-|s|^2)|f'(s)|, \max_{t\in \bar{D}(z,r)} (1-|t|^2)|g'(t)|\}$$
$$\times \max\{\max_{s\in \bar{D}(z,r)} (1-|s|^2)|f'(s)|, \max_{t\in \bar{D}(z,r)} (1-|t|^2)|g'(t)|\}.$$

Since f and g are in the Bloch space, there is a constant $B>0$ such that

$$\max\{\max_{s\in \bar{D}(z,r)} (1-|s|^2)|f'(s)|, \max_{t\in \bar{D}(z,r)} (1-|t|^2)|g'(t)|\} \leqslant B.$$

Therefore

$$\int_{D(z,r)} |f(w)-f(z)||g(w)-g(z)||k_2|^2\,dA(w)$$
$$\leqslant BC^2 \min\{\max_{s\in \bar{D}(z,r)} (1-|s|^2)|f'(s)|, \max_{t\in \bar{D}(z,r)} (1-|t|^2)|g'(t)|\}.$$

By the hypothesis we obtain

$$\lim_{|z|\to 1} \int_{D(z,r)} |f(w)-f(z)||g(w)-g(z)||k_2|^2\,dA(w)=0,$$

completing the proof.

LEMMA 4. *If f and g are in the Bloch space and for all* $0<r<1$

$$\min\{\max_{s\in \bar{D}(2,r)}(1-|s|^2)|f'(s)|,\ \max_{t\in \bar{D}(z,r)}(1-|t|^2)|g'(t)|\}\to 0,$$

as $|z|\to 1$, *then*

$$\lim_{|z|\to 1}\int_D |f(w)-f(z)||g(w)-g(z)||k_2|^2\,dA(w)=0.$$

Proof. Now we estimate the following integral for fixed $0<r<1$:

$$\begin{aligned}\int_{D\backslash D(z,r)} &|f(w)-f(z)||g(w)-g(z)||k_2|^2\,dA\\ &\leqslant\left(\int_{D\backslash D(z,r)}|f(w)-f(z)|^2\,|g(w)-g(z)|^2\,|k_2|^2\,dA\right)^{1/2}\\ &\quad\times\left(\int_{D\backslash D(z,r)}|k_2|^2\,dA\right)^{1/2}.\end{aligned}$$

It follows from [2, Theorem 2] that there is a constant $C>0$ such that

$$\begin{aligned}\left(\int_D|f(w)-f(z)|^4\,|k_2|^2\,dA\right)^{1/4}\leqslant C\,\|f\|_\beta\\ \leqslant\left(\int_\beta|f(w)-f(z)|^4\,|k_z|^2\,dA\right)^{1/4}\\ \times\left(\int_\beta|g(w)-g(z)|^4\,|k_2|^2\right)^{1/4}(1-r^2)^{1/2}\end{aligned}\tag{3}$$

and

$$\left(\int_D|g(w)-g(z)|^4\,|k_2|^2\,dA\right)^{1/4}\simeq C\,\|g\|_\beta.$$

For any $\varepsilon>0$ we may choose

$$\delta=\frac{\varepsilon^2}{(1+r)\,C^4(\|f\|_\beta+1)^2\,(\|g\|_\beta+1)^2}.$$

If $1-r<\delta$ the inequality (3) implies

$$\begin{aligned}\int_{D\setminus D(z,r)} &|f(w)-f(z)|\,|g(w)-g(z)|\,|k_z|^2\,dA\\ &\leqslant C^2\,\|f\|_\beta\,\|g\|_\beta\,(1-r^2)^{1/2}\\ &< C\,\|f\|_\beta\,\|g\|_\beta\,\frac{\varepsilon}{C(\|f\|_\beta+1)(\|g\|_\beta+1)}\\ &<\varepsilon.\end{aligned}$$

Thus

$$\begin{aligned}\int_D &|f(w)-f(z)|\,|g(w)-g(z)|\,|k_z|^2\,dA\\ &\leqslant \int_{D\setminus D(z,r)} |f(w)-f(z)|\,|g(w)-g(z)|\,|k_z|^2\,dA\\ &\quad+\int_{D(z,r)} |f(w)-f(z)|\,|g(w)-g(z)|\,|k_z|^2\,dA\\ &\leqslant \varepsilon+\int_{D(z,r)} |f(w)-f(z)|\,|g(w)-g(z)|\,|k_z|^2\,dA\end{aligned}$$

if $1-r<\delta$. Lemma 3 says

$$\lim_{|z|\to 1}\int_{D(z,r)} |f(w)-f(z)|\,|g(w)-g(z)|\,|k_z|^2\,dA(w)=0$$

for fixed $0<r<1$. The above inequality gives

$$\overline{\lim_{|z|\to 1}}\int_D |f(w)-f(z)|\,|g(w)-g(z)|\,|k_z|^2\,dA(w)\leqslant\varepsilon.$$

So

$$\lim_{|z|\to 1}\int_D |f(w)-f(z)|\,|g(w)-g(z)|\,|k_z|^2\,dA(w)=0$$

since ε is arbitrary. The proof is finished.

We state the following lemma which is proved in [2] and will be used in the proof of Lemma 6.

LEMMA 5. *Let*

$$K=\sup\left\{\int_D |1-z\alpha|^{-6/5}\,(1-|\alpha|)^{-3/5}\,dA(\alpha)\colon z\in D\right\}.$$

Then $K<\infty$.

The following lemma will be used twice in the proof of Theorem 2.

LEMMA 6. *There is a constant $C>0$ such that*

$$\int_D \frac{|f(w)-f(z)\|\,g(w)-g(z)|}{|1-zw|^2\sqrt{1-|w|^2}}\,dA(w) \leqslant \frac{C}{\sqrt{1-|z|^2}}\left[\int_D |f(w)-f(z)\|\,g(w)-g(z)\|\,k_2|^2\,dA(w)\right]^{1/12}$$

for all $z\in D$.

Proof. Fix $z\in D$, and make the change of variables by $\lambda=\phi_z(w)$ to get

$$\begin{aligned}
\int_D &\frac{|f(w)-f(z)\|\,g(w)-g(z)|}{|1-zw|^2\sqrt{1-|w|^2}}\,dA(w)\\
&=\frac{1}{\sqrt{1-|z|^2}}\left[\int_D \frac{|f\circ\phi_2(\lambda)-f(z)\|\,g\circ\phi_2(\lambda)-g(z)|}{|1-\bar z\lambda|\sqrt{1-|\lambda|^2}}\,dA(\lambda)\right]\\
&\leqslant\frac{1}{\sqrt{1-|z|^2}}\left[\int_D |f\circ\phi_2(\lambda)-f(z)|^\delta\,|g\circ\phi(\lambda)-g(z)|^\delta\,dA(\lambda)\right]^{1/6}\\
&\quad\times\left[\int_D |1-\bar z\lambda|^{-6/5}(1-|\lambda|^2)^{-3/5}\,dA(\lambda)\right]^{5/6}.
\end{aligned}$$

It follows from Lemma 5 that

$$\begin{aligned}
\int_D &\frac{|f(w)-f(z)\|\,g(w)-g(z)|}{|1-zw^2\sqrt{1-|w|^2}}\,dA(w)\\
&\leqslant\frac{1}{\sqrt{1-|z|^2}}\left[\int_D |f\circ\phi_2(\lambda)-f(z)|^\delta\,|g\circ\phi_2(\lambda)-g(z)|^\delta\,dA(\lambda)\right]^{1/6}K^{5/6}\\
&\leqslant\frac{1}{\sqrt{1-|z|^2}}\left[\int_D |f\circ\phi_2(\lambda)-f(z)|\,|g\circ\phi_2(\lambda)-g(z)|\,dA(\lambda)\right]^{1/12}K^{5/6}\\
&\quad\times\left[\int_D |f\circ\phi_2(\lambda)-f(z)|^{11}\,|g\circ\phi_2(\lambda)-g(z)|^{11}\,dA(\lambda)\right]^{1/12}
\end{aligned}$$

(by Cauchy–Schwarz inequality)

$$\begin{aligned}
&\leqslant\frac{1}{\sqrt{1-|z|^2}}\left[\int_D |f\circ\phi_2(\lambda)-f(z)|\,|g\circ\phi_2(\lambda)-g(z)|\,dA(\lambda)\right]^{12}K^{5/6}\\
&\quad\times\left[\int_D |f\circ\phi_2(\lambda)-f(z)|^{22}\,dA(\lambda)\right]^{1/66}\\
&\quad\times\left[\int_D |g\circ\phi_2(\lambda)-g(z)|^{22}\,dA(\lambda)\right]^{1/66}
\end{aligned}$$

(by Cauchy–Schwarz inequality)

$$\leqslant \frac{1}{\sqrt{1-|z|^2}} \left[\int_D |f \circ \phi_2(\lambda) - f(z)| \, |g \circ \phi_2(\lambda) - g(z)| \, dA(\lambda) \right]^{1/12} \times K^{5/6} \, \| f \|_\beta^{1/\delta} \, \| g \|_\beta^{1/\delta}$$

(this inequality comes from [2, Theorem 2])

$$\leqslant \frac{C}{\sqrt{1-|z|^2}} \left[\int_D |f \circ \phi_2(\lambda) - f(z)| \, |g \circ \phi_2(\lambda) - g(z)| \, dA(\lambda) \right]^{1/12}.$$

The proof is complete.

THEOREM 2. *If f and g are in the Bloch space and*

$$\lim_{|z| \to 1} \int_D |f(w) - f(z)| \, |g(w) - g(z)| \, |k_2|^2 \, dA(w) = 0,$$

then $H_{\bar{f}}^* H_{\bar{g}}$ is compact.

Proof. For any $h \in L_a^2(D)$ and $z \in D$ we have

$$\begin{aligned}
(H_{\bar{f}}^* H_{\bar{g}} h)(z) &= \frac{1}{1-|z|^2} \langle H_{\bar{f}}^* H_{\bar{g}} h, k_z \rangle \\
&= \frac{1}{1-|z|^2} \langle H_{\bar{g}} h, H_{\bar{f}} k_z \rangle \\
&= \frac{1}{1-|z|^2} \langle (\bar{g} - \bar{g}(z)) \, h, (\bar{f} - \bar{f}(z)) \, k_2 \rangle \\
&= \int_D \frac{(f(w) - f(z))(\bar{g}(w) - \bar{g}(z))}{(1 - z\bar{w})^2} h(w) \, dA(w).
\end{aligned}$$

It is obvious that for fixed $0 < r < 1$ the operator S_r defined by

$$S_r h(z) = \int_D \frac{(f(w) - f(z))(\bar{g}(w) - \bar{g}(z))}{(1 - z\bar{w})^2} h(w) \, \chi_{D(0,r)}(z) \, dA(w)$$

is a compact operator from $L_a^2(D)$ to $L^2(D)$. In fact, S_r is a Hilbert–Schmidt operator because

$$\frac{(f(w) - f(z))(\bar{g}(w) - \bar{g}(z))}{(1 - z\bar{w})^2} \chi_{D(0,r)}(z)$$

is in $L^2(D \times D)$.

For any $h \in L_a^2(D)$ and $z \in D$ we have

$$
\begin{aligned}
&|H_f^* H_{\bar g} h(z) - S_r h(z)| \\
&\quad \leqslant \int_D \frac{\chi_{D\setminus rD}(w) |f(w)-f(z)| |g(w)-g(z)| |h(w)|}{|1-z\bar w|^2}\, dA(w) \\
&\quad \leqslant \left[\int_D \frac{\chi_{D\setminus rD}(z) |f(w)-f(z)| |g(w)-g(z)|}{|1-z\bar w|^2 \sqrt{1-|w|^2}}\, dA(w)\right]^{1/2} \\
&\qquad \times \left[\int_D \frac{\chi_{D\setminus rD}(z) |f(w)-f(z)| |g(w)-g(z)| \sqrt{1-|w|^2}}{|1-z\bar w|^2} |h(w)|^2\, dA(w)\right]^{1/2}.
\end{aligned} \tag{4}
$$

Combining (4) and Lemma 6 gives

$$
\begin{aligned}
&\|(H_f^* H_{\bar g} - S_r) h\|^2 \\
&\quad = \int_D |[(H_f^* H_{\bar g} - S_r) h](z)|^2\, dA(z) \\
&\quad \leqslant \int_D \left[\int_D \frac{\chi_{D\setminus rD}(z) |f(w)-f(z)| |g(w)-g(z)|}{|1-z\bar w|^2 \sqrt{1-|w|^2}}\, dA(w)\right] \\
&\qquad \times \left[\int_D \frac{|f(w)-f(z)| |g(w)-g(z)| - \sqrt{1-|w|^2}}{|1-\bar z w|^2} |h(w)|^2\, dA(w)\right] dA(z)
\end{aligned}
$$

(by Cauchy–Schwarz inequality)

$$
\begin{aligned}
&\quad \leqslant \int_{D\setminus rD} \frac{C}{\sqrt{1-|z|^2}} \left[\int_D |f(w)-f(z)| |g(w)-g(z)| |k_2|^2\, dA(w)\right]^{1/12} \\
&\qquad \times \left[\int_D \int_D \frac{|f(w)-f(z)| |g(w)-g(z)| \sqrt{1-|w|^2}}{|1-\bar z w|^2} |h(w)|^2\, dA(w)\right] dA(z)
\end{aligned}
$$

(by Lemma 6)

$$
\begin{aligned}
&\quad \leqslant C \sup_{z \in D\setminus rD} \left[\int_D |f(w)-f(z)| |g(w)-g(z)| |k_2|^2\, dA(w)\right]^{1/2} \\
&\qquad \times \int_D \int_D \frac{|f(w)-f(z)| |g(w)-g(z)| \sqrt{1-|w|^2}}{|1-\bar z w|^2 \sqrt{1-|z|^2}} |h(w)|^2\, dA(w)\, dA(z)
\end{aligned}
$$

(by Lemma 6 again)

$$\leqslant C \sup_{u \in D \setminus rD} \left[\int_D |f(w)-f(u)\| g(w)-g(u)\| k_u|^2 \, dA(w) \right]^{1/2}$$

$$\times \int_D |h(w)|^2 \frac{C\sqrt{1-|w|^2}}{\sqrt{1-|w|^2}} \left[\int_D |f(w)-f(z)\| g(w)$$

$$-g(z)\| k_w|^2 \, dA(z) \right]^{1/12} dA(w)$$

$$\leqslant C \sup_{u \in D \setminus rD} \left[\int_D |f(w)-f(u)\| g(w)-g(u)\| k_u|^2 \, dA(w) \right]^{1/12}$$

$$\times C \sup_{u \in D} \left[\int_D |f(w)-f(u)\| g(w)-g(u)\| k_u|^2 \, dA(w) \right]^{1/12}.$$

It is easy to verify that

$$C \sup_{u \in D} \left[\int_D |f(w)-f(u)\| g(w)-g(u)\| k_u|^2 \, dA(w) \right]^{1/2} = \mu^2$$

is bounded since f and g are in Bloch space. Thus

$$\| H_f^* H_{\bar{g}} - S_r \| \leqslant \mu \, C \sup_{u \in D \setminus rD} \left[\int_D |f(w)-f(u)\| g(w)-g(u)\| k_u|^2 \, dA(w) \right]^{1/24}.$$

Since

$$\lim_{|z| \to 1} \int_D |f(w)-f(z)\| g(w)-g(z)\| k_z|^2 \, dA(w) = 0,$$

we have

$$\lim_{r \to 1} \| H_f^* H_{\bar{g}} - S_r | = 0;$$

so $H_f^* H_{\bar{g}}$ is compact since S_r is compact for any $0 < r < 1$, completing the proof.

Now we turn to the proof of the main results of this section. Although Theorem 3 is a corollary of Theorem 4, we give a proof of Theorem 3 by combining with the lemmas and Theorem 2.

THEOREM 3. *If f and g are in $H^\infty(D)$ and either f or g is constant on each Gleason part $P(m)$ of $\mathcal{M}$, then $H_f^* H_{\bar{g}}$ is compact.*

Proof. From Theorem 2 it suffices to prove

$$\lim_{|z| \to 1} \int_D |f(w)-f(z)\| g(w)-g(z)\| k_z|^2 \, dA(w) = 0. \tag{5}$$

Combining Lemmas 2 and 3 with Lemma 4 implies that the above equation (5) holds. So $H_f^* H_{\bar{g}}$ is compact.

THEOREM 4. *If f and g are bounded harmonic functions on D and for each thin part $P(m)$ of $\mathcal{M}$ either $f|_{P(m)} \in H^\infty|_{P(m)}$ or $g|_{P(m)} \in H^\infty|_{P(m)}$, then*

(a) *$H_f^* H_g$ is compact;*

(f) $\lim_{|z| \to 1} \min\{(1-|z|^2)|(\partial f/\partial \bar{z})(z)|, (1-|z|^2)|(\partial g/\partial \bar{z})(z)|\} = 0$.

Proof. Since f and g are bounded harmonic functions on D, there are functions f_1, f_2, g_1, and g_2 in the Bloch space such that $f = f_1 + \bar{f}_2$ and $g = g_1 + \bar{g}_2$. Thus

$$H_f^* H_g = H_{\bar{f}_2}^* H_{\bar{g}_2}$$

and $(\partial f/\partial \bar{z})(z) = \bar{f}_2'(z)$, $(\partial g/\partial \bar{z})(z) = \bar{g}_2'(z)$.
Combining Lemmas 2–4 with Theorem 2 shows that it is sufficient to prove that for fixed $0 < r < 1$

$$\lim_{|z| \to 1} \min\{\max_{s \in \bar{D}(z, r)} (1-|s|^2)|f_2'(s)|, \max_{t \in \bar{D}(z, r)} (1-|t|^2)|g_2'(t)|\} = 0. \tag{6}$$

Suppose that (6) does not hold. There are points $\{z_n\} \subset D$ and $\varepsilon > 0$ such that

$$\min\{\max_{s \in \bar{D}(z_n, r)} (1-|s|^2)|f_2'(s)|, \max_{t \in \bar{D}(z_n, r)} (1-|t|^2)|g_2'(t)|\} \geqslant \varepsilon$$

and $\{z_n\}$ has no accumulation points in D. Since there is a thin subsequence of $\{z_n\}$, we may assume that $\{z_n\}$ is thin. Let m be in the closure $\{z_n\}$ in $\mathcal{M}$. Without loss of generality we may assume $f|_{P(m)} \in H^\infty|_{P(m)}$ and $\{z_n\}$ converges to m. Let $\{w_n\}$ be points in D satisfying

$$\begin{aligned} &1.\ w_n \in D(z_n, r) \\ &2.\ (1-|w_n|^2)|f'(w_n)| = \max_{s \in \bar{D}(z_n, r)} (1-|s|^2)\,|f_2'(s)|. \end{aligned} \tag{7}$$

Since $\mathcal{M}$ is compact, there is a subnet of $\{w_n\}$ converging to some m_1. For convenience we may assume that $\{w_n\}$ converges to m_1. Since $\rho(z_n, w_n) \leqslant r$, then m_1 is in $P(m)$. Since $f|_{P(m)} \in H^\infty|_{P(m)}$, we have

$$f|_{P(m_1)} \in H^*|_{P(m_1)}.$$

Thus

$$\frac{\partial}{\partial \bar{z}} f \circ L_{m_1}(0) = 0.$$

On the other hand

$$\begin{aligned}\frac{\partial}{\partial \bar{z}} f \circ L_{m_1}(0) &= \lim_{w_n \to m_1} \frac{\partial}{\partial \bar{z}} f \circ L_{w_n}(0) \\ &= \lim_{w_n \to m_1} \frac{\partial}{\partial \bar{z}} (f_1 \circ L_{w_n} + \bar{f}_2 \circ L_{w_n})(0) \\ &= \lim_{w_n \to m_1} (1 - |w_n| 2) \bar{f}'_2(w_n).\end{aligned}$$

This contradicts (7). The proof is finished.

2

In this section we first consider compactness of $H_{\bar{f}}^* H_{\bar{g}}$ for f and g in $H^\infty(D)$ and make use of the maps L_m to turn the compactness of $H_{\bar{f}}^* H_{\bar{g}}$ into the condition

$$H^*_{\bar{f} \circ L_m} H_{\bar{g} \circ L_m} = 0.$$

The following Lemma 7 is the partial result of [6, Theorem X.2.5].

LEMMA 7. *Let $m(z) = L_m(z)$ be in $P(m)$ for some z in D. Then there is a constant c, $|c| = 1$ such that*

$$L_{m(z)}(cw) = L_m \circ \phi_z(w).$$

Proof. It follows from H2 that L_m and $L_{m(z)}$ are one-to-one analytic maps from D onto the Gleason part $P(m)$. Thus $L^{-1}_{m(z)} \circ L_m \circ \phi_z \colon D \to D$ is an onto, one-to-one, analytic function, and $L^{-1}_{m(z)} \circ L_m \circ \phi_z(0) = 0$. It is well known that there is a constant c such that $|c| = 1$ and

$$L^{-1}_{m(z)} \circ L_m \circ \phi_z(w) = cw.$$

So $L_{m(z)}(cw) = L_m \circ \phi_z(w)$.

Although Proposition 1 is a corollary of Proposition 2, we give its proof since the proof is also interesting.

PROPOSITION 1. *If $H_{\bar{f}}^* H_{\bar{g}}$ is compact for f and g in $H^\infty(D)$, then*

$$H^*_{\bar{f} \circ L_m} H_{\bar{g} \circ L_m} = 0$$

for all m in $\mathcal{M} \backslash D$.

Proof. For any $m \in \mathcal{M} \backslash D$ there is a net $\{z_n\} \subset D$ converging to m (Corona Theorem), so

$$f \circ L_{2_n}(w) - f \circ L_{2_n}(0) \to f \circ L_m(w) - f \circ L_m(0) \qquad \text{pointwise,}$$
$$g \circ L_{2_n}(w) - g \circ L_{2_n}(0) \to g \circ L_m(w) - g \circ L_m(0) \qquad \text{pointwise.}$$

In addition $f \circ L_m(z)$ and $g \circ L_m(z)$ are bounded on D. So for any bounded analytic function h

$$\lim_{2_n \to m} \int_D h(f \circ L_{2_n} - f \circ L_{2_n}(0))(\bar{g} \circ L_{2_n} - \bar{g} \circ L_{2_n}(0))\, dA$$
$$= \int_D h(f \circ L_m - f \circ L_m(0))(\bar{g} \circ L_m - \bar{g} \circ L_m(0))\, dA.$$

On the other hand,

$$\int_D h(f \circ L_{z_n} - f \circ L_{z_n}(0))(\bar{g} \circ L_{z_n} - \bar{g} \circ L_{z_n}(0))\, dA$$
$$= \int_D (f - f \circ L_{z_n}(0))(\bar{g} - \bar{g} \circ L_{z_n}(0)) |k_{z_n}|^2\, h \circ L_{z_n}\, dA$$
$$= \langle H_{\bar{g}} h \circ L_{z_n} k_{z_n}, H_{\bar{f}} k_{z_n} \rangle = \langle h \circ L_{z_n} k_{z_n}, H_{\bar{g}}^* H_{\bar{f}} k_{z_n} \rangle. \tag{8}$$

Since $H_{\bar{f}}^* H_{\bar{g}}$ is compact

$$\| H_{\bar{g}}^* H_{\bar{f}} k_{z_n} \| \to 0 \qquad \text{as} \quad z_n \to m.$$

Therefore

$$|\langle h \circ L_{z_n} k_{z_n}, H_{\bar{g}}^* H_{\bar{f}} k_{z_n} \rangle| \leqslant \| h \circ L_{z_n} k_{z_n} \| \, \| H_{\bar{g}}^* H_{\bar{f}} k_{z_n} \|$$
$$\leqslant \| h \|_m \, \| H_{\bar{g}}^* H_{\bar{f}} k_{z_n} \| \to 0 \text{ as } z_n \to m. \tag{9}$$

Combining (8) and (9) we get that

$$\lim_{z_n \to m} \int_D h(f \circ L_{z_n} - f \circ L_{z_n}(0))(\bar{g} \circ L_{z_n} - \bar{g} \circ L_{z_n}(0))\, dA = 0.$$

This implies that

$$\int_D h(f \circ L_m - f \circ L_m(0))(\bar{g} \circ L_m - \bar{g} \circ L_m(0))\, dA = 0 \tag{10}$$

for all bounded analytic functions h. Replacing m by $m(z)$ in (10) we obtain

$$\int_D h(f \circ L_{m(z)} - f \circ L_{m(z)}(0))(\bar{g} \circ L_{m(z)} - \bar{g} \circ L_{m(z)}(0))\, dA = 0.$$

The above equality combined with Lemma 7 implies

$$\int_D h(f\circ L_m\circ\phi_z - f\circ L_m(z))(\bar{g}\circ L_m\circ\phi_2 - \bar{g}\circ L_m(z))\,dA = 0.$$

Changing the variables by $\lambda = \phi_z(w)$ gives

$$\int_D h\circ\phi_z(f\circ L_m - f\circ L_m(0))(\bar{g}\circ L_m - \bar{g}\circ L_m(0))|k_2|^2\,dA = 0.$$

We may subsitute $h\circ\phi_z$ for h to change the above equality into

$$\int_D h(f\circ L_m - f\circ L_m(0))(\bar{g}\circ L_m - \bar{g}\circ L_m(0))|k_2|^2\,dA = 0.$$

Thus

$$\langle h, H^*_{\bar{g}\circ L_m} H_{f\circ L_m} k_2\rangle = 0.$$

We know that $H^\infty(D)$ is dense in $L^2_a(D)$. So

$$H^*_{f\circ L_m} H_{\bar{g}\circ L_m} k_2 = 0,$$

which implies that

$$H^*_{f\circ L_m} H_{\bar{g}\circ L_m} = 0.$$

Before we generalize the above proposition for bounded harmonic functions f and g, we need the following lemma which is proved in [1].

LEMMA 8. *Let ϕ be a Mobius transformation from D onto D and define an operator U_ϕ on $L^2(D)$ by*

$$U_\phi g(z) = g[\phi(z)][\phi'(z)].$$

Then

(a) *U_ϕ is unitary,*
(b) *$PU_\phi = U_\phi P$.*

PROPOSITION 2. *If f and g are bounded harmonic functions and $H^*_f H_g$ is compact, then*

$$H^*_{f\circ l_m} H_{g\circ L_m} = 0$$

for all m in $\mathscr{M}\backslash D$.

Proof. Since f and g are harmonic functions, there are Bloch functions

f_1, f_2, g_1, and g_2 such that $f=f_1+\bar{f}_2$ and $g=g_1+\bar{g}_2$. For any m in $\mathcal{M}\backslash D$ there is a net $\{z_n\}$ converging to m. Then

$$\begin{aligned} &\frac{\partial}{\partial \bar{z}}[f\circ L_{z_n}(z)] \to \frac{\partial}{\partial \bar{z}}[f\circ L_m(z)] \qquad \text{pointwise} \\ &\frac{\partial}{\partial \bar{z}}[g\circ L_{z_n}(z)] \to \frac{\partial}{\partial \bar{z}}[g\circ L_m(z)] \qquad \text{pointwise.} \end{aligned} \tag{11}$$

Since $f\circ L_m$ and $g\circ L_m$ are bounded harmonic on D, there are Bloch functions f_3, f_4, g_3, and g_4 such that

$$f\circ L_m = f_3+\bar{f}_4 \qquad \text{and} \qquad \frac{\partial}{\partial \bar{z}}(f\circ L_m) = \bar{f}_4',$$

$$g\circ L_m = g_3+\bar{g}_4 \qquad \text{and} \qquad \frac{\partial}{\partial \bar{z}}(g\circ L_m) = \bar{g}_4'.$$

So

$$\begin{aligned} \bar{f}_4(w)-\bar{f}_4(0) &= \int_0^1 \frac{\partial}{\partial \bar{z}}[f\circ L_m(tw)]\,dt \\ \bar{g}_4(w)-\bar{g}_4(0) &= \int_0^1 \frac{\partial}{\partial \bar{z}}[g\circ L_m(tw)]\,dt. \end{aligned} \tag{12}$$

Now

$$\langle H^*_{fd_m} H_{gd_m} k_0, hk_0\rangle = \int_D (f_4(w)-f_4(0))(\bar{g}_4(w)-\bar{g}_4(0))\, h dA(w). \tag{13}$$

For fixed h in $H^\infty(D)$ and any $\varepsilon>0$ there is a r_0 in $(0,1)$ such that if $1>r>r_0$ then

$$\int_{D\backslash rD} |f_4(w)-f_4(0)|\,|g_4(w)-g_4(0)|\,|h|\,dA(w)<\varepsilon$$

$$\int_{D\backslash rD} |f_2\circ\phi_{z_n}(w)-f_2\circ\phi_{z_n}(0)|\,|g_2\circ\phi_{z_n}(w)-g_2\circ\phi_{z_n}(0)|\,|h|\,dA(w)<\varepsilon$$

for all z_n in D. Combining (11) with (12) implies

$$\begin{aligned} &\left|\int_D [f_4(w)-f_4(0)][\bar{g}_4(w)-\bar{g}_4(0)]\,\bar{h} dA(w)\right| \\ &= \left|\int_D |w|^2 \int_0^1\int_0^1 \frac{\partial}{\partial \bar{z}} g\circ L_m(tw)\frac{\partial}{\partial z}\bar{f}\circ L_m(sw)\,dt\,ds\,\bar{h}(w)\,dA(w)\right| \\ &\leqslant \varepsilon + \varlimsup_{z_n\to m}\left|\int_{rD} |w|^2 \int_0^1\int_0^1 \frac{\partial}{\partial \bar{z}} g\circ L_{z_n}(tw)\frac{\partial}{\partial z}\bar{f}\circ L_{z_n}(sw)\,dt\,ds\,\bar{h}(w)\,dA(w)\right| \end{aligned}$$

(by Fatou's lemma and (11))

$$\leqslant 2\varepsilon + \overline{\lim_{z_n \to m}} \left| \int_D [f_2 \circ L_{z_n}(w) - f_2(z_n)][\bar{g}_2 \circ L_{z_n}(w) - \bar{g}_2(z_n)]\, dA(w) \right|$$

$$\leqslant 2\varepsilon + \overline{\lim_{z_n \to m}} \; |\langle H_f^* H_g l_{z_n}, h \circ \phi_{z_n} k_{z_n} \rangle|$$

$$\leqslant 2\varepsilon.$$

Since ε is arbitrary

$$\langle H^*_{f \circ L_m} H_{g \circ L_m} k_0, hk_0 \rangle = 0.$$

Substituting $m(z)$ for m we have

$$\langle H^*_{f \circ L_{m(z)}} H_{g \circ L_{m(z)}} k_0, hk_0 \rangle = 0.$$

Using Lemmas 7 and 8 we obtain

$$\langle H^*_{f \circ L_m} H_{g \circ L_m} k_z, h \circ \phi_z k_z \rangle = 0.$$

This implies

$$H^*_{f \circ L_m} H_{g \circ L_m} = 0.$$

LEMMA 9. *Suppose that f and g are bounded harmonic functions. If $H_f^* H_g = 0$ then for all $Z \in D$ and $\xi \in \partial D$*

$$H^*_{f \circ \phi_z} H_{g \circ \phi_z} = 0 \qquad \text{and} \qquad H^*_{f_\xi} H_{g_\xi} = 0,$$

where $f_\xi(w) = f(\xi w)$.

Proof. Let ϕ be a Mobius function mapping D onto D. From Lemma 8 it is easy to verify that

$$U_\phi H_f^* H_g U_\phi^* = H^*_{f \circ \phi} H_{g \circ \phi}.$$

So this implies the lemma if $\phi(w)$ is replaced by $\phi_z(w)$ or ξw, respectively.

Before going on we comment on some facts that will be used in the proof of Theorem 5. Suppose that $f = f_1 + \bar{f}_2$ and $g = g_1 + \bar{g}_2$, where f_i and g_1 are in Bloch space and Hardy space H^2. If $H_f^* H_g = 0$, then

$$\int_D [f_2 \circ \phi_z(w) - f_2(z)][\bar{g}_2 \circ \phi_z(w) - \bar{g}_2(z)]\, dA(w) = 0.$$

This is equivalent to

$$\int_D f_2(w)\, \bar{g}_2(w) |k_z|^2 \, dA(w) = f_2(z)\, \bar{g}_2(z). \tag{14}$$

Replacing f_2 and g_2 by $f_2 \circ \phi_\lambda$ and $g_2 \circ \phi_\lambda$ or $f_{2\xi}$ and $g_{2\xi}$, respectively, by Lemma 9 we have

$$\int_D f_2 \circ \phi_\lambda(w)\, \bar{g}_2 \circ \phi_\lambda(w) |k_z|^2\, dA(w) = f_2 \circ \phi_\lambda(z)\, \bar{g}_2 \circ \phi_\lambda(z)$$

and

$$\int_D f_2(\xi w)\, \bar{g}_2(\xi w) |k_z|^2\, dA(w) = f_2(\xi z)\, \bar{g}_2(\xi z),$$

where $z \in D$ and $\xi \in \partial D$.

We state the following lemma which is the special case of [1, Proposition 10.2].

LEMMA 10. *Let f be a continuous function on the closed unit disk $\bar{D}$. Then the following are equivalent*:

(a) *f is harmonic on D*;

(b) *for each z in D*

$$f(z) = \int_D |k_z|^2 f(\xi)\, dA(\xi).$$

THEOREM 5. *Suppose f and g are bounded harmonic functions on D. If $H_f^* H_g = 0$, then either f or g is in $H^\infty(D)$.*

Proof. Let $f = f_1 + \bar{f}_2$ and $g = g_1 + \bar{g}_2$ where f_i and g_i are in the Bloch space and H^2. Then $H_f^* H_g = 0$ implies

$$H_{\bar{f}_2}^* H_{\bar{g}_2} = 0.$$

The remark after Lemma 9 gives

$$\int_D f_2 \circ \phi_\lambda(w)\, \bar{g}_2 \circ \phi_\lambda(w) |k_z|^2\, dA(w) = f_2 \circ \phi_\lambda(z)\, \bar{g}_2 \circ \phi_\lambda(z)$$

and

$$\int_D f_2(\xi w)\, \bar{g}_2(\xi w) |k_z|^2\, dA(w) = f_2(\xi z)\, \bar{g}_2(\xi z).$$

Set

$$G(z) = \int_{\xi \in \partial D} f_2(\xi z)\, \bar{g}_2(\xi z)\, d\theta/2\pi$$

and suppose $f_2 \circ \phi_\lambda(w) = \sum_{n=0}^{\infty} a_n(\lambda)\, w^n$ and $g_2 \circ \phi_\lambda(w) = \sum_{n=0}^{\infty} b_n(\lambda)\, w^n$. Then $\sum_{n=0}^{\infty} |a_n(\lambda)|^2 < \infty$ and $\sum_{n=0}^{\infty} |b_n(\lambda)|^2 < \infty$. Thus

$$G(z) = \sum^{\infty} a_n(0)\, \bar{b}_n(0) |z|^{2n}.$$

Since $\sum_{n=0}^{\infty} a_n(0)\, \bar{b}_n(0) |z|^{2n}$ converges unformly on $\bar{D}$, the function G is continuous. By (14) we get

$$\int_D G(w) |k_z(w)|^2 \, dA(w) = G(z).$$

It follows from Lemma 10 that $G(z)$ is harmonic. Let Δ denote the Laplace operator. It is easy to verify that

$$\Delta G(z) = 4 \sum_{n=0}^{\infty} n^2 a_n(0)\, \bar{b}_n(0) |z|^{2(n-1)}.$$

So $\Delta G(z) = 0$ implies that $a_n(0)\, \bar{b}_n(0) = 0, n > 1$. Similarly we can prove that $a_n(z)\, \bar{b}_n(z) = 0, n > 1$. Now we consider only $a_1(z)\, b_1(z) = 0$. Without loss of generality we may assume that there are points $\{w_n\}$ in $D(0, r)$ for some $0 < r < 1$ and $\{w_n\}$ has at least one accumulation in $D(0, r)$ such that $a_1(w_n) = 0$ for all n. In fact $a_1(z) = [f_2 \circ \phi_z]'(0) = (1 - |z|^2) f_2'(z)$. Thus $f_2'(w_n) = 0$. Therefore $f_2'(w) = 0$ for all w in D. This means that f is constant. The proof is complete.

Considering Toeplitz operators on the Bergman space we interpret the theorem to mean that $T_{\bar{f}}^* T_g = T_{\bar{f}g}$ for bounded harmonic functions f and g iff either f or g is in $H^\infty(D)$. On the Hardy space the above result is true for all f and g in $L^\infty(\partial D)$. But on the Bergman space we do not know when $T_{\bar{f}}^* T_g = T_{\bar{f}g}$ is true for f and g in $L^\infty(\bar{D})$.

Now we turn to the proof of the main theorem in the section.

THEOREM 6. *Suppose f and g are bounded harmonic functions on D. If $H_{\bar{f}}^* H_g$ is compact, then either $f \circ L_m$ or $g \circ L_m$ is in $H^\infty(D)$ for m in $\mathcal{M} \backslash D$.*

Proof. Proposition 2 says that

$$H_{\bar{f} \circ L_m}^* H_{g \circ L_m} = 0$$

for all m in $\mathcal{M} \backslash D$ if $H_{\bar{f}}^* H_g$ is compact. It follows from Theorem 5 that either $f \circ L_m$ or $g \circ L_m$ is in $H^\infty(D)$.

Theorem 6 gives that either f or g is in $H^\infty(D)|_{P(m)}$ on each thin part $P(m)$ since $b \circ L_m(z) = z$ for some b in $H^\infty(D)$. So far we have proved that (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) $\Leftrightarrow$ (f). To complete the section we will prove

(d) $\Leftrightarrow$ (e). The theorem of Axler and Shields makes the proof of the following theorem possible.

THEOREM 7. *Let f and g be bounded harmonic fuctions on D. The following are equivalent*:

(c) *For each thin part $P(m)$ except D, either $f|_{P(m)} \in H^\infty(D)|_{P(m)}$ or $g|_{P(m)} \in H^\infty(D)|_{P(m)}$*;

(e) $H^\infty(D)[f] \cap H^\infty(D)[g] \subset \{u \in C(\mathcal{M}): u|_{P(m)} \in H^\infty(D)$ *for each thin part* $P(m)\}$.

Proof. That (c) implies (e) is obvious. Now we prove that (e) implies (c). Let m in $\mathcal{M}\backslash D$ and $P(m)$ be an analytic disk. That $H^\infty(D)[f] \cap H^\infty(D)[g] \subset \{u \in C(\mathcal{M}): u|_{P(m)} \in H^\infty(D)$ for thin part $P(m)\}$ means that

$$H^\infty(D)[f] \cap H^\infty(D)[g]|_{P(m)} \subset H^\infty(D)|_{P(m)}. \tag{15}$$

In fact $H^\infty(D)[f] \cap H^\infty(D)[g] \circ L_m = H^\infty \circ L_m(D)(f \circ L_m) \cap H^\infty L_m(D)$ $[g \circ L_m]$ and $H^\infty \circ L_m(D) = H^\infty$ since $P(m)$ is thin. Thus

$$H^\infty(D)[f] \cap H^\infty(D)[g] \circ L_m = H^\infty(D)[f \circ L_m] \cap H^\infty(D)[g \circ L_m]. \tag{16}$$

The theorem of Axler and Shields says that if u and v are bounded harmonic but not analytic on D, then $H^\infty(D) + C(\bar{D}) \subset H^\infty(D)[u] \cap H^\infty(D)[v]$. Equations (15) and (16) imply

$$H^\infty(D)[f \circ L_m] \cap H^\infty(D)[g \circ L_m] \subset H^\infty(D).$$

Thus either $f \circ L_m$ or $g \circ L_m$ is in $H^\infty(D)$. We have finished the proof.

3

(1) From the proof of Theorem 3 we see that the theorem is also valid if D is replaced by the unit ball B_n in C^n. It is natural to ask if Theorem 6 is true on B_n. But no one knows what the Gleason parts of the maximal ideal space of $H^\infty(B_n)$ look like. In fact whether the Corona Theorem is valid on $\mathcal{M}(H^\infty(B_n))$ is unknown.

(2) Looking at our proof carefully we observe that the main results in the paper are also valid on the weighted Bergman space using the following Proposition 3 instead of Lemma 5 in the above process.

PROPOSITION 3. *Let α, $\beta > -1$. Then if $[\frac{1}{2} - (\alpha/2 - \beta/q)] < 1/p < 1$ and $1/q + 1/p = 1$, there is an $M > 0$ such that for all z in D*

$$\int_D \frac{(1 - |w|^2)^{P(\alpha/2 - \beta/q)}}{|1 - \bar{z}w|^2 (1 - |w|^2)^{P/2}} dA(w) < M.$$

Indeed we can obtain the result on the weigted Bergman space analogous to that on the Bergman space in [9] by means of Proposition 3. Before we state the following theorem, we define the weighted Bergman space and $\mathrm{VMO}_\partial(D)$. The weighted Bergman space $A_{2\alpha}(\alpha > -1)$ is defined by

$$\{f : f \text{ is analytic on the unit disk } D \text{ and } \int |f(z)|^2 (1 - |z|^2)^\alpha \, dA(z) < \infty\},$$

and $\mathrm{VMO}_\partial(D)$ is the following set

$$\{f \in L^1(D) : \int |\tilde{f}(z) - f \circ \phi_z(u)| \, dA(u) \to 0 \text{ as } |z| \to 1\},$$

where $\tilde{f}(z) = \int f(u) |k_z(u)|^2 \, dA(u)$. Roughly speaking $\mathrm{VMO}_\partial(D)$ is the space of intergrable functions on D with vanishing mean oscillation near the boundary of D.

THEOREM 8. *Let $\alpha > -1$ and f be in $L^\infty(D)$. Then H_f and $H_{\bar{f}}$ are compact on weighted Bergman space $A_{2\alpha}$ iff f is in* $\mathrm{VMO}_\partial(D)$.

(3) From many recent results on Hankel operators and Toeplitz operators on the Bergman space it seems more natural to deal with the Toeplitz operators and Hankel operators with bounded harmonic symbols than with the symbols in $L^\infty(D)$.

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